

A Brief Summary of Laurent Series

18.1

Let f be a holomorphic function on an annulus $D = \{z \in \mathbb{C} : r < |z - z_0| < R\}$. Then f can be expanded into a series of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n},$$

where both series converge on D and uniformly on every subannulus of D concentric with D . Let C be any simple closed contour in D concentric with containing z_0 and oriented once in the counterclockwise direction.

Then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = 0, \pm 1, \pm 2, \dots$$

We call $\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ the Laurent Series of f at z_0 .

Theorem: Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ be series such that

- $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges on $\{z \in \mathbb{C} : |z - z_0| < R\}$,
- $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ converges on $\{z \in \mathbb{C} : |z - z_0| > r\}$,
- $r < R$.

Then there exists a unique holomorphic function F on $D = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ such that the Laurent series of F on D is

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}, \quad r < |z - z_0| < R.$$

Proof: Assume $z_0 = 0$. Let $S = \frac{1}{z_0}$. Then

$$\sum_{n=1}^{\infty} a_n S^n$$

Converges on $\left\{ S \in \mathbb{C} : |S| < \frac{1}{r} \right\}$. Let

$$H(S) = \sum_{n=1}^{\infty} a_n S^n, |S| < \frac{1}{r}.$$

Then H is holomorphic on $\left\{ S \in \mathbb{C} : |S| < \frac{1}{r} \right\}$. So the function $h(z) = H\left(\frac{1}{z}\right)$ is holomorphic on $|z| > r$.

$$\text{and } h(z) = \sum_{n=1}^{\infty} a_n z^{-n}, |z| > r.$$

Now the function $g(z) = \sum_{n=1}^{\infty} a_n z^n, |z| < R$, is holomorphic. as the function given by

$$p(z) = g(z) + h(z)$$

is holomorphic on $D = \left\{ z \in \mathbb{C} : r < |z| < R \right\}$.
as $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n}$ is the Laurent series of p in D . To do this, let C be a simple closed contour in D at z_0 . To do this, let C be a simple closed contour in D enclosing 0 and oriented once in the counterclockwise direction. Then for all j

$$\begin{aligned}
 &= 0, \pm 1, \pm 2, \dots \\
 \frac{1}{2\pi i} \int_C \frac{p(z)}{z^{j+1}} dz &= \frac{1}{2\pi i} \int_C \sum_{n=1}^{\infty} a_n z^{n-j-1} dz \\
 &= \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} a_n z^{n-j-1} dz \\
 &\quad \text{[Diagram of a contour C in the complex plane, showing a small circle around the origin and a large circle enclosing it. The contour is oriented clockwise. A point z_0 is marked on the real axis to the left of the origin. The region D is shaded in light blue. The contour C is drawn as a closed loop in the first quadrant, starting from the origin, going up, then right, then down, then left, returning to the origin. The radius of the outer circle is labeled r. The radius of the inner circle is labeled R. The distance from the origin to z_0 is labeled r_0. The angle between the positive real axis and the line segment from the origin to z_0 is labeled theta_0.]} \\
 &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{n=0}^{\infty} a_n z^{n-j-1} dz \\
 &= \frac{1}{2\pi i} a_j 2\pi i = a_j.
 \end{aligned}$$

Example: Find the Laurent series of $f(z) = \frac{z^2 - 2z + 3}{z-2}$

Or $\{z \in \mathbb{C} : |z-1| > 1\}$.

Solution: $z_0 = 1$, $r = 1$, $R = \infty$. $\therefore f$ is holomorphic on $\{z \in \mathbb{C} : 0 < |z-1| < \infty\}$.

$$\text{Now } \frac{1}{z-2} = \frac{1}{z-1-1} = \frac{1}{z-1} \cdot \frac{1}{1 - \frac{1}{z-1}} \\ \approx \frac{1}{z-1} \sum_{n=0}^{\infty} \frac{1}{(z-1)^n} = \sum_{n=0}^{\infty} \frac{1}{(z-1)^{n+1}}.$$

But $z^2 - 2z + 3 = z^2 - 2z + 1 + 2 = (z-1)^2 + 2$.

$$\therefore f(z) \\ = \left\{ (z-1)^2 + 1 + \frac{1}{z-1} + \frac{1}{(z-1)^2} + \dots \right\} + \left\{ \frac{2}{z-1} + \frac{2}{(z-1)^2} + \dots \right\} \\ = (z-1)^2 + 1 + \sum_{n=1}^{\infty} \frac{3}{(z-1)^n}, |z-1| > 1.$$

Example: Find the Laurent series of $e^{\frac{1}{z}}$ at $z=0$.

Solution: $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$
 $= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} z^{-n}, z \neq 0$.

Why Laurent Series?

Isolated Singularities

18.4.

Definition : Let $w = f(z)$ be a complex-valued function.

Let $z_0 \in \mathbb{C}$ be $\exists \{ f \text{ is not holomorphic at } z_0 \}$
 $\{ f \text{ is holomorphic on some punctured disk of } z_0 \}$



Then we say that z_0 is an isolated singularity of f .

Let z_0 be an isolated singularity of f . Then f is
 holomorphic on a punctured disk

$\{ z \in \mathbb{C} : 0 < |z - z_0| < R \}$. or
 Three Possibilities on $\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z^{-n})$

- If $a_{-n} = 0$ for $n=1, 2, \dots$, then we call z_0 a removable singularity of f .
- If $a_{-m} \neq 0$ for some $m \in \mathbb{N}$ and $a_{-n} = 0$ for all $n > m$, then we call z_0 a pole of order m of f . A pole of order 1 is called a simple pole.
- If $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$, then we call z_0 an essential singularity of f .

Example : Classify the isolated singularities

of $f(z) = e^{\frac{1}{z}}$, $z \neq 0$

Solution : 0 is the only isolated singularity of f .

Also, $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$
 the Laurent series of $e^{\frac{1}{z}}$

Now note that $a_{-n} = \frac{1}{n!} \neq 0$ for all $n \in \mathbb{N}$.